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DETERMINATION OF STRESSES AND DEFORMATIONS OF AIRCRAFT PROPELLERS

By Friedrich Seewald

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DETERMINATION OF STRESSES AND DEFORMATIONS OF AIRCRAFT PROPELLERS*

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The propeller is probably one of the most highly stressed parts of an aircraft. It is therefore surprising that the strength of aircraft propellers has hitherto received so little attention. While our knowledge of the strength of airplane wings has been considerably increased in recent years, relatively little is yet known regarding the stresses undergone by aircraft propellers. It appears important therefore to give this subject some attention. The incentive to the following investigation was furnished by a series of accidents caused by propeller failures in flight. A large proportion of these failures occurred with a type of propeller which had a peculiar curving shape. This propeller could not be made to fail on the test stand, even with considerable overloading, although a blade root of one of the propellers used was already damaged.

At first thought this fact appears rather remarkable, since the forces generated in stand tests at increased revolution speeds are much greater than in flight. In particular, the thrust forces on the stand are a multiple of those in the air. If, however, the action of the forces on a propeller blade is more closely considered, it is immediately recognized that the shape of the blade has a preponderant effect on the bending moments. The centrifugal force acts lengthwise of the blade, while the aerodynamic force acts perpendicularly to it. Under present conditions of propeller operation, the centrifugal force is one hundred or more times as great as the aerodynamic force. If the blade is not quite straight, the points of application of the centrifugal forces to the in-

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dividual blade elements do not lie in a straight line and consequently generate bending moments which tend to stretch the blade. Since the centrifugal force is preponderantly great in comparison with the other forces, even slight deviations from the straight line are important. Of course it does not matter whether the deflections were originally present or were produced elastically by the loading. This is a well-known fact, and patents have already been granted for a propeller with blades intended to be so shaped that the centrifugal and aerodynamic moments will just offset one another. The blade axis could then be regarded as a flexible line on which all the forces act instead of on a solid body. Great caution must be exercised, however, with such propellers, as illustrated by the above-mentioned accidents. It is not sufficient for one to have only a qualitative knowledge of the effect of the forces, but he must also be able to calculate the magnitude of the bending moments with consideration of the shape and elastic properties. The only scientific publication known to me on the strength investigation of aircraft propellers is the work by Reissner, "Ueber die Biegungsbeanspruchung von Luftschrauben und die entlastende Wirkung der Zentrifugalkraft," Technische Berichte, Vol. II, No. 2, 1917, page 315. In this work the elastic deflections and the centrifugal moments due to these deflections are determined for a propeller which is conceived from the first as a straight untwisted bar, for which a definite distribution of the cross sections and inertia moments, is assumed. Proceeding from similar assumptions, Wilhelm Hoff developed a graphic method for determining the strength of propellers which he explained in his lectures at the Berlin Polytechnic Institute. This work has not yet been published (1927). Based on these works, C. Jansen undertook an investigation on how the strength calculation of a propeller is affected by twisting the blade sections (D.V.L. Report No. 50, Zeitschrift für Flugtechnik und Motorluftschiffahrt, 1925, page 87). The assumptions concerning the shape of the blade axis and the distribution of the inertia forces along the blade are essentially the same as in Reissner's work. Since, however, for the above-mentioned reasons, even slight deviations from the preliminary assumptions entail considerable changes in the bending moments, it is obviously important, in propeller investigations, to introduce the true form into the calculation, especially because, in most propellers and for constructional reasons, the original blade axis deviates more from a straight line than is ascribable to the elastic deflection.

In what follows, we must therefore regard the aircraft-propeller blade as an originally bent and twisted bar, which is still further elastically deformed under an added load. The accurate solution of this problem encounters difficulties out of all proportion to the attainable results. The whole problem, however, can be greatly simplified by remembering that only the excessive magnitude of the centrifugal force, which acts approximately parallel to the longitudinal axis of the blade, makes it at all necessary to consider the very slight original curvature. Hence, in determining the conditions of equilibrium, we will consider the total curvature (embracing that elastically produced and that originally present) only in so far as it involves the resolution of the static forces, which are so large in comparison with the others that even a relatively small component has some effect. If the centrifugal force, which is approximately parallel to the axis of the blade, is resolved into a force lengthwise of the blade and another force perpendicular to the blade, the latter component is only a small fraction of the centrifugal force. Since, however, the other forces acting in this direction are very small in comparison with the centrifugal force, even this small component plays a part. Conversely, it would be useless to consider any small component of the already small aerodynamic force, if it should fall in the direction of the centrifugal force. Moreover, since the deviations of the blade axis from a straight line are very small (in extreme cases not over 1/30 of the blade length), the blade may be treated, with respect to the forces perpendicular or nearly perpendicular to the blade axis, as if the blade axis were straight.

Even in calculating the internal stresses, the curvature may be disregarded, and the formulas of elementary mechanics (for example, $\sigma = M_y/J$) may be considered valid. The resulting error is negligible since the radii of curvature are multiples of the cross-sectional dimensions. Note that we are here dealing with cross sections which are asymmetric with respect to the axis passing through the center of gravity and that, even in this point, the assumptions of elementary mechanics are therefore not strictly fulfilled. However, since the deviations from symmetry are not excessive, we may expect to obtain sufficiently accurate results.

CONDITIONS OF EQUILIBRIUM

The blade axis is the line passing through the centers of gravity of the individual sections. It may be any curve in space, but deviates only a little from a straight line. We shall establish this curve by the following system of coordinates. The x axis is placed in the radial direction so as to conform as closely as possible to the blade axis, the y axis in the direction of flight and the z axis perpendicular to both. The positive directions are so chosen as to form a right-hand system. The centrifugal force then acts very closely in the direction of the x axis. The centrifugal force developed by a blade element of length Δs is designated by ΔC. The aerodynamic force is assumed to act in the yz plane and is resolved into the components p_y and p_z. We then have:

$$\int_0^R p_y \, dx = \frac{S}{a} \quad \text{and} \quad \int_0^R p_z \, dx \, x = \frac{M}{a},$$

in which S denotes the thrust, M the torque of the engine and a the number of blades.

Two projections of the blade axis are shown in Figures 1 and 2. The manner of resolution then follows directly from the above statements, especially the justification for designating the forces in the directions y and z as transverse forces. The component in the direction of the blade axis is designated by ΔN and the components in the directions of the y and z axes by ΔQ_y and ΔQ_z, respectively. The magnitudes of these forces are:

$$\Delta N = \frac{\Delta C}{\cos(sx)} = \sim \Delta C$$

$$\Delta Q_y = p_y \, \Delta x - \Delta C \cos(sy)$$

$$\Delta Q_z = p_z \, \Delta x - \Delta C \cos(sz).$$

The cross section of the blade at the point $x = x_1$ is acted on by the normal force

$$N = \sum_{x_1}^{R} \Delta C \tag{1}$$

and by the transverse forces:

$$Q_y = \sum_{x_1}^{R} p_y \, \Delta x - \sum_{x_1}^{R} \Delta C \cos(sy) \tag{2}$$

$$Q_z = \sum_{x_1}^{R} p_z \, \Delta x - \sum_{x_1}^{R} \Delta C \cos(sz) \tag{3}$$

The components of the bending moment are denoted by M_y and M_z. The bending moment M_y tends to produce rotation about the y axis. It is positive when the resulting curvature of the blade axis is concave in the positive z direction. Corresponding statements apply to M_z. The components of the bending moment are:

$$M_y = \int_{x_1}^{R} p_y (x - x_1) \, dx - \int_{x_1}^{R} \frac{\Delta C}{\Delta x} (y - y_1) \, dx \tag{4}$$

$$M_z = \int_{x_1}^{R} p_z (x - x_1) \, dx - \int_{x_1}^{R} \frac{\Delta C}{\Delta x} (z - z_1) \, dx \tag{5}$$

The torsional moment is due partly to the fact that the aerodynamic force acting on a blade element does not pass through the center of gravity. Shifting the aerodynamic force toward the center of gravity would require a corresponding moment, which may be designated by M_{od}. The torsional moment is also due in part to the curvature of the blade axis. Hence,

$$\begin{aligned} M_d = & \int_{x_1}^{R} \frac{dM_o}{dx} [\cos(sx) - \cos(s_1 x)] \, dx - \cos(s_1 x) \int_{x_1}^{R} p_y (z - z_1) dx + \\ & + \cos(s_1 x) \int_{x_1}^{R} p_z (y - y_1) dx + \cos(s_1 y) \int_{x_1}^{R} \left[p_y (x - x_1) - \frac{\Delta C}{\Delta x} (y - y_1) \right] dx - \\ & - \cos(s_1 z) \int_{x_1}^{R} \left[p_z (x - x_1) - \frac{\Delta C}{\Delta x} (z - z_1) \right] dx \end{aligned} \tag{6}$$

The last two expressions represent the components of the bending moments M_y and M_z, in the direction of the blade axis. Since the torque is small in comparison with the bending moment, these components must be taken into consideration, while the torque components in the direction of the bending moment may be disregarded.

The bending moments are the most important in the strength investigation and must be resolved in the direction of the principal cross-sectional axes. The determination of the principal axes is quite troublesome. For practical purposes it suffices to take one direction parallel to the blade chord and the other perpendicular to it. The comparison of the computed principal directions for several blade sections with the assumed ones yielded a difference of only about two degrees, which is of no practical importance. There is naturally no objection to using the actual principal axes in the further consideration of the subject. They will be designated by I and II, as in Figure 3. The angle formed by the projection of axis I with the z axis will be designated by α. This angle is called the pitch angle and is defined by the expression $\tan \alpha = H/2\pi r$, where H is the pitch of the propeller. On resolving M_y and M_z in the direction of the principal axes I and II, we obtain:

$$M_I = M_y \cos \alpha + M_z \sin \alpha$$

$$M_{II} = - M_y \sin \alpha + M_z \cos \alpha.$$

The elastic curvatures thus produced are designated by k_I and k_{II} corresponding to the inertia moments J_I and J_{II}, and we then have:

$$k_I = \frac{M_I}{E\ J_I} = \frac{1}{E\ J_I} (M_y \cos \alpha + M_z \sin \alpha) \tag{7}$$

$$k_{II} = \frac{M_{II}}{E\ J_{II}} = \frac{1}{E\ J_{II}} (- M_y \sin \alpha + M_z \cos \alpha) \tag{8}$$

If ϑ denotes the angle of elastic distortion of the blade and J_d the torsional strength, the torque will be given by the expression

$$\frac{d\vartheta}{dx} = \frac{M_d}{G\ J_d} \tag{9}$$

The curvatures k and the torque ϑ here represent simple elastic deformations, which must be expressed by the coordinates of the blade. Hence the coordinates must be divided, on the one hand, into those representing the original shape and, on the other hand, into those representing the elastic deformations. The shape of the unstressed blade axis is denoted by the coordinates

$$x_o,\ y_o,\ z_o \text{ and } \alpha_o$$

and the corresponding elastic deformations by ξ, η,

ζ and ϑ. The variation in the length of the blade is essentially represented by ξ and is so small that it may be disregarded.

The final coordinates of a point on the blade axis in the stressed condition are therefore

$$x = x_0, \; y = y_0 + \eta, \; z = z_0 + \zeta \text{ and } \alpha = \alpha_0 + \vartheta.$$

The elastic curvatures k_I and k_{II}, expressed by these coordinates, are:

$$k_I = \frac{d^2\eta}{dx^2}\cos(\alpha_0 + \vartheta) + \frac{d^2\zeta}{dx^2}\sin(\alpha_0 + \vartheta) = \frac{M_I}{E\ J_I}$$

$$k_{II} = -\frac{d^2\eta}{dx^2}\sin(\alpha_0 + \vartheta) + \frac{d^2\zeta}{dx^2}\cos(\alpha_0 + \vartheta) = \frac{M_{II}}{EJ_{II}}.$$

In equations (7) and (8), the quantities M_I and M_{II} are expressed by M_y and M_z. By introducing these values, we obtain

$$k_I = \frac{d^2\eta}{dx^2}\cos(\alpha_0 + \vartheta) + \frac{d^2\zeta}{dx^2}\sin(\alpha_0 + \vartheta) =$$

$$= \frac{M_y\cos(\alpha_0 + \vartheta)}{E\ J_I} + \frac{M_z\sin(\alpha_0 + \vartheta)}{E\ J_I} \tag{10}$$

$$k_{II} = -\frac{d^2\eta}{dx^2}\sin(\alpha_0 + \vartheta) + \frac{d^2\zeta}{dx^2}\cos(\alpha_0 + \vartheta) = -$$

$$-\frac{M_y\sin(\alpha_0 + \vartheta)}{E\ J_{II}} + \frac{M_z\cos(\alpha_0 + \vartheta)}{E\ J_{II}} \tag{11}$$

To these we must also add the equation:

$$\frac{d\vartheta}{dx} = \frac{M_d}{G\ J_d} \tag{12}$$

For convenience the value of M_d is not written out in equation (6). The moments M_y and M_z are expressed in equations (4) and (5) by the coordinates of the blade axis and by the forces acting on the blade. If, on the one hand, the coordinates y and z in the equations are divided, as mentioned above, into the components y_0 and z_0

which determine the original form and, on the other hand, into the components η and ζ which denote the elastic deformations, equations (4) and (5) then become

$$M_y = \int_{x_1}^{R} p_y(x - x_1)\, dx - \int_{x_1}^{R} \frac{\Delta C}{\Delta x} (y_o - y_{o1})\, dx - - \int_{x_1}^{R} \frac{\Delta C}{\Delta x} (\eta - \eta_1)\, dx \qquad (13)$$

$$M_z = \int_{x_1}^{R} p_z(x - x_1)\, dx - \int_{x_1}^{R} \frac{\Delta C}{\Delta x} (z_o - z_{o1})\, dx - - \int_{x_1}^{R} \frac{\Delta C}{\Delta x} (\zeta - \zeta_1)\, dx \qquad (14)$$

These values must be introduced into equations (10) to (12), in order to obtain the conditional equations for the elastic deformations in terms of the blade form and of the acting forces. In equations (13) and (14) the first two integrals represent the moments which would be produced by the forces acting on the unbent blade. These can be calculated directly. The last integrals in the equations for M_y and M_z contain the change in the moments due to the change in the shape of the whole blade under stress and the consequent variation in the points of application of the forces. The nature and magnitude of these deformations can be represented by exponential series and write the expressions:

$$\eta = a_o + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots$$

$$\zeta = b_o + b_1 x + b_2 x^2 + b_3 x^3 + \ldots\ldots$$

$$\vartheta = c_o + c_1 x + c_2 x^2 + c_3 x^3 + \ldots\ldots$$

We calculate with these expressions as if they were known quantities, leaving the constants a, b, and c to be subsequently determined so that the actual deformations will be represented as accurately as possible by these series. If we introduce these series into equations (10) to (14), we can calculate all unknown quantities as functions of a, b, and c. If we compute numerically the quantities in these equations for any values of x, we obtain, for each value of x, three equations in which only the quantities a, b, and c occur as unknowns. If, for example, we wish to consider the first three terms of each series, we must compute equations (10) to (14) for three values of x and thus obtain nine equations for the nine unknowns. In this general form the solution would still be very troublesome, because all the unknowns occur in every equation. This difficulty is remedied as follows. We first assume that $\vartheta = 0$, leaving only α_0 in the angular functions, i.e., the original twist. Furthermore, we put $k_{II} = 0$. This is justified by the fact that, in propeller sections, the inertia moment J_{II} is always much larger than J_I. It will appear later that both ϑ and k_{II} are actually so small that they can be put equal to zero in the first approximation. Equation (11) then becomes

$$k_{II} = -\frac{d^2\eta}{dx^2} \sin \alpha_0 + \frac{d^2\zeta}{dx^2} \cos \alpha_0 = 0 \qquad (15)$$

On multiplying this equation by $\sin \alpha_0$ and subtracting it from equation (10) multiplied by $\cos \alpha_0$, in which likewise $\vartheta = 0$, we obtain

$$\frac{d^2\eta}{dx^2} = (M_y \cos^2 \alpha_0 + M_z \sin \alpha \cos \alpha_0) \frac{1}{E\, J_I} \qquad (16)$$

The only other quantity now appearing on the left side is the second differential coefficient of η. The right side, however, contains, in addition to η, as follows from equation (5), also ζ in M_z. By assuming $k_{II} = 0$, we can also express ζ as a function of η and therefore by α. Equation (15) then yields

$$\frac{d^2\zeta}{dx^2} = \frac{d^2\eta}{dx^2} \tan \alpha.$$

As already explained, α is the pitch angle and consequently,

$$\tan \alpha = \frac{H}{2 \pi x}.$$

For most propellers the pitch H is approximately constant over the whole blade. Where considerable deviations occur, $\tan \alpha$ should be expressed approximately by a corresponding function of x and the procedure should then be exactly as follows:

$$\frac{d^2\zeta}{dx^2} = \frac{d^2\eta}{dx^2} \frac{H}{2 \pi x} \qquad (17)$$

For η we developed the expression

$$\eta = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots$$

We can immediately put $a_0 = a_1 = 0$, because the blade may be regarded as fixed at the hub, so that η and $d\eta/dx$ must vanish for $x = 0$. Hence,

$$\frac{d^2\eta}{dx^2} = 2\ a_2 + 6\ a_3 x + 12\ a_4 x^2 + 20\ a_5 x^3 + \ldots\ldots$$

By introducing this value into equation (17), we obtain

$$\frac{d^2\zeta}{dx^2} = (2\ a_2 + 6\ a_3 x + 12\ a_4 x^2 + 20\ a_5 x^3 + \ldots)\ \frac{H}{2\ \pi\ x}.$$

ζ can now be computed by integration.

$$\frac{d\zeta}{dx} = (2\ a_2 \ln x + 6\ a_3 x + 6\ a_4 x^2 + + \frac{20}{3}\ a_5 x^3 + \ldots\ldots)\ \frac{H}{2\pi} + C_1$$

$d\zeta/dx$ must equal zero for $x = 0$, which is possible only when $a_2 = C_1 = 0$. By another integration, we obtain

$$\zeta = (3\ a_3 x^2 + 2\ a_4 x^3 + \frac{5}{3}\ a_5 x^4 + \ldots\ldots)\ \frac{H}{2\pi} \qquad (17a)$$

Now all the quantities in equation (16) can be expressed by η. From this equation we then calculate an approximate value for η and simultaneously also for ζ. We can then compute all the bending moments. At first these values are only approximate. In the numerical example, however, it will be found that the assumptions $k_{II} = 0$ and $\vartheta = 0$ are so accurately fulfilled, that the values calculated on the basis of this assumption may be regarded as final. Should this be otherwise in particular instances, the values may be corrected by introducing the calculated approximate values ϑ and k_{II} and making the calculation again.

Hence the problem is to determine the constants a of the series

$$\eta = a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots\ldots$$

in such a way that equation (16) is satisfied for as many values of x as there are terms in the series for η.

We break off the series for η with the third term and find, as the first condition, that all moments become zero at the blade tip, including also the curvature $d^2\eta/dx^2$. The other two conditions are determined as follows. If an assumed function is correct, and the bending moments on the right side of equation (16) are calculated, these bending moments necessitate certain definite deflections. If these are calculated, they must equal the assumed values of η at every point. This condition cannot be realized at all points, but at as many points as there are terms in the series. Since we have not yet had over two constants, we can satisfy this condition for only two points. We arbitrarily stipulate that the deflection at the blade tip and for $x = 0.6\ R$ shall be the correct one. We have then represented the deflection by an approximate function which agrees with the actual bending line under the following conditions. At the hub it is fixed, while it has the correct deflection for $x = 0.6$. At the blade tip it has the correct deflection and the correct curvature. It may be assumed that such a function, which satisfies five boundary conditions, will not deviate greatly from the correct curve. As conditions for the last two constants, it was stipulated that the deflections and not the curvature should agree at two points. If the curvature had been chosen, then two conditions for the constants would have been obtained. The result would still have been very inaccurate, principally due to the fact that, on the right side, only the inertia moments were introduced at the two places. As a result of the double integration of M/EJ, however, the magnitudes of the inertia moments and cross sections were indirectly taken into consideration at all points by means of the deflections.

NUMERICAL EXAMPLE

The process of calculation will be illustrated by an example, taking the propeller described in the introduction. It will be found that the stresses in flight, especially at high speeds and therefore at low thrust, are considerably greater than at the same revolution speed on the stand. The propeller used in the tests was of this type with a diameter $D = 2.45$ m (8.04 ft.) and a pitch $H = 1.15$ m (3.77 ft.). It was tested on the stand at 1450 r.p.m. and yielded a thrust of 325 kg (716.5 lb.) with an engine torque of 47 m-kg (340 lb.-ft.). The stresses were determined under these conditions. The propeller was run at a speed of 1700 r.p.m. on the stand without injury.

The propeller-blade sections were 10 cm (3.94 in.) apart. In Figure 4 the cross-sectional area is plotted against the radius. The c.g. of each measured section was found and the coordinates y_0 and z_0 of the blade axis were determined. The blade-axis curves are shown in Figure 5. Figures 6a and 6b show, respectively, the sine and cosine of the angle made by the blade-section chord with the plane of the propeller. Figures 7 and 8 show, respectively, the inertia moments J_I and J_{II} at each point of the blade (J_{II} on ten times the scale of J_I). All these quantities are plotted against the nondimensional abscissa $\xi = x/R$, where x represents the distance of a cross section from the center of the hub.

Equation (16), when the above-given series and the substitution $\xi = x/R$ are introduced for η and ξ with consideration of equations (13) and (14), takes the form

$$6\, a_3 \xi + 12\, a_4 R^2 \xi^2 + 20\, a_5 R^3 \xi^3 = \frac{\cos^2\alpha}{E\, J_I} \left\{ R^2 \int_{\xi=\xi_1}^{\xi=1} p_y (\xi - \xi_1)\, d\xi - \right.$$

$$\left. - \int_{\xi=\xi_1}^{\xi=1} \frac{\Delta C}{\Delta \xi} (y_0 - y_{01})\, d\xi \right\} - \frac{\cos^2\alpha}{E\, J_I} \int_{\xi=\xi_1}^{\xi=1} \frac{\Delta C}{\Delta \xi} [R^3\, a_3 (\xi^3 - \xi_1{}^3) +$$

$$+ R^4 a_4 (\xi^4 - \xi_1{}^4) + R^5 a_5 (\xi^5 - \xi_1{}^5)]\, d\xi$$

$$+ \frac{\sin\alpha\, \cos\alpha}{E\, J_I} \left\{ R^2 \int_{\xi=\xi_1}^{\xi=1} p_z (\xi - \xi_1)\, d\xi - \int_{\xi=\xi_1}^{\xi=1} \frac{\Delta C}{\Delta \xi} (z_0 - z_{01})\, d\xi \right\}$$

$$- \frac{\sin\alpha\, \cos\alpha}{E\, J_I} \int_{\xi=\xi_1}^{\xi=1} \frac{\Delta C}{\Delta \xi} [0.55\, R^2 a_3 (\xi^2 - \xi_1{}^2) +$$

$$+ 0.366\, R^3 a_4 (\xi^3 - \xi_1{}^3) + 0.305\, R^4 a_5 (\xi^4 - \xi_1{}^4)]\, d\xi \qquad (18)$$

In order to determine the deflection itself, it is necessary to integrate twice, whereby it must be remembered that α and J are functions of ξ. All the terms on the right side can now be computed. However, the integrations cannot all be computed since α, J and the centrifugal force at each point are not all given in terms of ξ, but take a course which can be determined only by measurements. All these expressions are therefore obtained by summation. The distribution of the aerodynamic forces is approximately known. For the strength calculation, it suffices to approximate them by a simple function. For example, the component forces which produce the thrust, may be assumed to have a distribution which is approximately represented by the function

$$p_y = k\,\xi^2 \sqrt{1 - \xi^2}$$

The constant k must be so determined that the sum over all the blades will give their total thrust. It must therefore be

$$\int_0^1 k\,\xi^2 \sqrt{1 - \xi^2}\, R\, d\xi = \frac{S}{a}$$

where a is the fluid coefficient, whence

$$k = \frac{16\ S}{\alpha\ \pi\ R}$$

The bending moment due to the thrust is therefore easily calculated. At the point $\xi = \xi_1$ it is

$$M_s = R^2 \int_{\xi=\xi_1}^{\xi=1} p_y (\xi - \xi_1)\, d\xi = R\, \frac{16\ S}{\pi\ n} \int_{\xi=\xi_1}^{\xi=1} \xi^2 \sqrt{1-\xi^2}\,(\xi-\xi_1)\, d\xi$$

After integrating and again replacing ξ_1 by ξ, we obtain

$$M_s = \frac{16\ S\ R}{\alpha\ \pi} \left[\frac{\sqrt{1 - \xi^2}}{120} (6\ \xi^4 + 7\ \xi^2 + 16) - \right.$$

$$\left. - \frac{1}{8} \xi \left(\frac{\pi}{2} - \text{arc sin}\, \xi \right) \right]$$

Hence the moment exerted by the thrust is

$$M_s = \frac{16\ S\ R}{\alpha\ \pi}\, f(\xi),$$

in which the bracketed portion of the preceding formula is replaced by $f(\xi)$. This function has the values:

$\xi =$	0	0.1	0.2	0.3	0.6	0.5	0.6	0.7
$f(\xi)=$	0.133	0.114	0.094	0.075	0.057	0.040	0.0255	0.0139

$\xi =$	0.8	0.9	1.0
$f(\xi) =$	0.0056	0.0011	0

The aerodynamic component acting in the plane of the propeller disk is represented accurately enough by the function

$$p_z = k_1 \; \xi \sqrt{1 - \xi^2}$$

We again determine k_I, so that the total moment exerted by p_z is equal to the measured or calculated engine torque.

$$\frac{M_m}{a} = \int_0^1 p_z \, R^2 \xi \, d\xi = k_1 \, R^2 \int_0^1 \xi^2 \sqrt{1 - \xi^2} \, d\xi = k_1 \, R^2 \frac{\pi}{16}$$

Hence

$$k_1 = \frac{M_m \; 16}{R^2 \; \pi \; a}$$

$$p_z = \frac{M_m \; 16}{R^2 \; \pi \; a} \; \xi \sqrt{1 - \xi^2}.$$

Then the bending moment due to this aerodynamic component at any point ξ_1 is

$$M = R^2 \int_{\xi=\xi_1}^{\xi=1} p_z (\xi - \xi_1) d\xi = \frac{M_m \; 16}{\pi \; a} \int_{\xi=\xi_1}^{\xi=1} \xi \sqrt{1 - \xi^2} (\xi - \xi_1) d\xi$$

$$= \frac{M_m \; 16}{\pi \; a} \left[\xi \frac{(2 \xi^2 - 1)}{8} - \frac{1}{8} \text{arc sin } \xi + \right.$$

$$\left. + \frac{\pi}{16} - \frac{\xi}{3} \sqrt{1 - \xi^2}^{\,3} \right] = \frac{M_m \; 16}{\pi \; a} \varphi(\xi).$$

For abbreviation the bracketed portion is replaced by $\varphi(\xi)$. It has the following values:

ξ =	0	0.1	0.2	0.3	0.4	0.5	0.6
$\varphi(\xi)$ =	0.1962	0.163	0.1312	0.106	0.073	0.0502	0.031

ξ =	0.7	0.8	0.9	1.0
$\varphi(\xi)$ =	0.0167	0.0044	0.0014	0

With these the bending moments due to the aerodynamic forces are determined. Their curves are plotted in Figures 9 and 10. We now have to determine the moments due to the centrifugal force and first those due to the original curvature. These bending moments are represented by the expressions:

$$\sum_{\xi=\xi_1}^{\xi=1} \frac{\Delta C}{\Delta \xi} (y_0 - y_{01}) \Delta \xi \quad \text{and} \quad \sum_{\xi=\xi_1}^{\xi=1} \frac{\Delta C}{\Delta \xi} (z_0 - z_{01}) \Delta \xi \; .$$

The quantities $y_0 - y_1$ and $z_0 - z_1$ can be read from Figure 5 for any value of ξ.

If we imagine the blade divided perpendicularly to the ξ axis into disks of the thickness $1 \text{ cm} = \Delta\xi$, then the centrifugal force per unit length is $\Delta C/R\,\Delta\xi$.

$$\frac{C}{R\,\Delta\,\xi} = F\,\frac{\gamma}{g}\,R\,\xi\,\omega^2$$

The specific gravity of the wood is assumed to be 0.8 or 800 kg/m³ (28 or 28250 cu.ft.). At 1450 r.p.m. ω^2 = 23000. In order to obtain $\Delta C/\Delta\xi$, we must therefore multiply the curve, which indicates the cross-sectional area, by the factor

$$\frac{\gamma}{g}\,\omega^2\,R\,\xi = \frac{800 \times 23000 \times 1.925}{9.81 \times 100^2}\,\xi = 229\,\xi \; .$$

This centrifugal force per unit length of the blade is plotted in Figure 11. For the calculation, the blade is divided into sections by ten planes perpendicular to the ξ axis at intervals of $\Delta\xi = 0.1$. In order to determine the centrifugal force of every such portion, we must determine the surface area of the curve $\Delta C/R\Delta\xi$ in Figure 11 between the limits ξ and $\xi + \Delta\xi$. We may conceive of such a portion with sufficient accuracy as a trapezoid, as shown in Figure 11. The point of application of this centrifugal force to the blade then lies at the value of ξ which corresponds to the c.g. of this surface strip. In what follows, however, it is more convenient to resolve this force in such a way that all the components are applied at $\xi = 0.1, 0.2$, etc. This resolution can be made by applying half of the area of the hatched rectangle in Figure 11 to each of the two points ξ and $\xi + w\ \xi$, and 2/3 of the area of the triangle to one point and the remaining 1/3 to the other point. The centrifugal forces for all the blade elements were thus determined.

$\xi =$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
	56	358	769	1081	1207	1225	1216	1140	949	602	110

For determining, at the point ξ, the bending moment due to these forces, we have the expression:

$$\sum_{\xi=\xi_n}^{\xi=1} \frac{\Delta C}{\Delta \xi} (y_o - y_{o1})\ \Delta\xi \quad \text{or} \quad \sum_{\xi=\xi_n}^{\xi=1} \frac{\Delta C}{\Delta \xi} (z_o - z_{o1})\ \Delta\xi$$

The simplest way to solve such an expression is by using the formula,

$$M_n = M_{n-1} + \Delta\ y_{on} \sum_{o}^{n=1} \Delta\ \xi\ \frac{\Delta C}{\Delta \xi}.$$

Δy_{on} and Δz_{on} are to be taken directly from Figure 5, where the original blade form is plotted. The moments $M_{C_{yo}}$ and $M_{C_{zo}}$ thus obtained are plotted in Figures 12 and 13. We are now in position to solve a portion of the right side of equation (18), namely, the expression:

$$\frac{\cos^2\alpha}{E\ J_I} [R^2 \int p_y(\xi - \xi_1)\ d\xi - \int \frac{\Delta C}{\Delta \xi} (y_o - y_{o1})\ d\xi] +$$

$$\frac{\sin\alpha\ \cos\alpha}{E\ J_I} [R^2 \int p_z(\xi - \xi_1) d\xi - \int \frac{\Delta C}{\Delta \xi} (z_o - z_{o1})\ d\xi]$$

The bracketed expressions have already been solved and represent the bending moments due to the action of the aerodynamic and centrifugal forces on the unbent blade. The angle α and the inertia moment J are measured at each point. $E = 100000$ kg/cm² (1420000 lb./sq.in.) was adopted as the elasticity modulus of wood. This whole expression has to be integrated twice, in order to obtain the deflection, which would be produced by the bracketed moments. A double integration is equivalent, however, to the production of a moment according to Mohr's theorem. The quantity represented by the above expression may therefore be taken as the load applied to a blade, from which the bending moment can be calculated. The evaluation can be made exactly as in the above determination of the bending moments. The result is plotted in Figure 14. The quantity, calculated as a bending moment, is designated by η_0 and represents the deflection produced by the above-calculated moments, when the moments remain constant during the deformation. The two still-unsolved expressions in equation (18)

$$\int \frac{\Delta C}{\Delta \xi} [R^3 \ a_3(\xi^3 - \xi_1^3) + R^4 \ a_4(\xi^4 - \xi_1^4) + R^5 \ a_5(\xi^5 - \xi_1^5)] \ d\xi$$

and $$\int \frac{\Delta C}{\Delta \xi} [0.55 \ R^2 \ a_3(\xi^2 - \xi_1^2) + 0.366 \ R^3 \ a_4(\xi^3 - \xi_1^3) + $$
$$+ 0.305 \ R^4 \ a_5(\xi^4 - \xi_1^4)] \ d\xi,$$

represent only the moments which are first produced by the elastic deformation and which overlap the above-calculated moments.

We will now continue with the terms containing the constant a_3. On extracting these terms we obtain the two expressions:

$$a_3 \ R^3 \int \frac{\Delta C}{\Delta \xi} (\xi^3 - \xi_1^3) \ d\xi$$

$$0.55 \ a_3 \ R^2 \int \frac{\Delta C}{\Delta \xi} (\xi^2 - \xi_1^2) \ d\xi.$$

Only known quantities stand under the integration sign. These can therefore be calculated by summation for any desired value of ξ_1, the calculation being made the same as for the previous quantities, as

$$\frac{\Delta C}{\Delta \xi} (y_0 - y_{01}) \ d\xi.$$

Both integrals represent functions of ξ_1, which can be directly computed and which we will represent by $\varphi(\xi)$ and $\psi(\xi)$. $a_3 \ \varphi(\xi)$ and $0.55 \ a_3 \ \psi(\xi)$ then represent the bending moments produced when the blade is bent into the form $\eta = a_3\,\xi^3$. By introducing these quantities into equation (18), we obtain

$$a_3 \left[\frac{\cos\alpha}{E\,J_I}\,\varphi(\xi) + 0.55\,\frac{\sin\alpha\,\cos\alpha}{E\,J_I}\,\psi(\xi)\right].$$

as the component of the bend produced by these moments. Again considering the bracketed expression as the load and having determined the bending moment, the latter is equivalent in magnitude to the deflection produced by the moments $\varphi(\xi)$ amd $0.55\,\psi(\xi)$ in the y direction. This deflection is designated by η_3. The procedure is the same with the terms a_4 and a_5, and the corresponding deflections are represented by η_4 and η_5.

All these deflections are plotted in Figure 15 on a magnified scale of 1000:1. The total deflection is then equal to the sum of all the individual deflections, so that the equation must be satisfied for every point ξ_1.

$$\eta = a_3\,R^3\,\xi_1{}^3 + a_4\,R^4\,\xi_1{}^4 + a_5\,R^5\,\xi_1{}^5 =$$

$$= \eta_0(\xi_1) + a_3\,\eta_3(\xi_1) + a_4\,\eta_4(\xi_1) + a_5\,\eta_5(\xi_1)$$

As already explained, we put ξ_1 on the left side, once $\xi_1 = 0.6$ and once $\xi_1 = 1$, while we introduce the calculated deflections η_0, η_3, η_4 and η_5 on the right side. Thus we obtain two equations for the constant a. For the third equation, as already mentioned, we choose the condition that, at the blade tip (that is, for $\xi_1 = 1$), all moments equal zero, so that $d^3\eta/d\xi^2$ must vanish. Thus we obtain the three equations:

$$1.225\times 1 + 12\,a_4\,1.225^2 + 20\,a_5\,1.225^3 = 0$$

$$1000(a_3\,1.225^3\times 0.6^3 + a_4\,1.225^4\times 0.6^4 + a_5\,1.225^5\times 0.6^5) =$$

$$= 1.473 - a_3\,250.16 - a_4\,254.7 - a_5\,247.6$$

$$1000(a_3\,1.225^3 + a_4\,1.225^4 + a_5\,1.225^5) = 13.126 -$$

$$- a_3\,854.9 - a_4\,924 - a_5\,1062.6.$$

These equations yield, for the a constants the values

$$a_3 = -0.017555$$

$$a_4 = +0.036033$$

$$a_5 = -0.01414$$

Hence the total deflection is

$$\eta = a_3 R^3 \xi^3 + a_4 R^4 \xi^4 + a_5 R^5 \xi^5 = -0.01755 \times 1.225^3 \xi^3 + \\ + 0.03603 \times 1.225^4 \xi^4 - 0.01414 \times 1.225^5 \xi^5$$

According to equation (17a)

$$\zeta = (3\ a_3\ \xi^2\ R^2 + 2\ a_4\ \xi^3\ R^3 + \frac{5}{3}\ a_5\ \xi^4\ R^4)\ \frac{H}{2\pi}$$

The quantities η and ζ are plotted in figures 16 and 17, from which it is obvious that the calculated deflection ζ is very small as compared with the deflection in the y direction. The shape of the propeller, after its deformation, is now known (fig. 18), and all moments can be calculated by equations (4), (5), and (6). The bending moments for $a_3 = 1$, $a_4 = 1$, and $a_5 = 1$ have already been calculated, so that we only need to multiply them by the computed values of a. The moments M_I and M_{II} can then be readily calculated from the individual expressions. Likewise the stresses can be directly calculated. The results are plotted in figures 19 to 22. The bending stresses due to M_2 were also calculated, though they are so small, due to the magnitude of J_2, that they play no part, all the more since the maximum stresses due to M_{II} occur in the fibers where the stresses due to M_I are almost zero. Since all the quantities are now known, the calculation of the shearing stresses can likewise be made as for ordinary beams. These stresses also are not very important. The torsional moment can be readily calculated from equation (6). It is so small, however, that it is of hardly any importance. It is plotted in figure 23. All these quantities are now calculated on the assumption that the curvature in the direction of the greater inertia moment and the elastic distortion of the blade are both negligibly small. In order to determine how far these conditions are satisfied, we will now calculate them. Since the magnitude of a distortion for any given cross section is very difficult to calculate, every cross section is replaced by an inscribed rectangle. The torsional rigidity of such a blade can then be determined according to St. Venant and is surely less than that of the actual blade. Likewise the modulus of shear G is assumed very low, $G = 8000$ kg/cm² (113800 lb./sq.in.). Under these assumptions, we have, at the propeller tip, a twist of $\vartheta = 0.00661$ or about 0.3°. This angle, however, is within the degree of experimental accuracy. The torsional angle might be somewhat greater for other blade forms. Nevertheless, it would hardly reach values which would affect the strength relations.

In order to test the other assumption $k_{II} = 0$, we will compute k_{II} and compare it with k_I. These quantities are calculated for a few values of ξ by forming the expressions M_I/EJ_I and M_{II}/EJ_{II}.

ξ	k_I	k_{II}
= 0.2	-1667.0	+29.2
= 0.4	+2620	-17.4
= 0.6	+5140	-36.8
= 0.8	+4430	-20.0

It is obvious that k_{II} is much smaller than k_I. At the most unfavorable point ($\xi = 0.4$), $k_{II} = 0.0175\ k_I$. It seems perfectly permissible, however, to disregard such a small quantity. This proves that the assumptions $k_2 = 0$ and $\vartheta = 0$ do not greatly impair the value of the final result, so that we may regard the calculated static data as correct.

It is obvious from the course of the bending moments and stresses that the moments due to the centrifugal forces preponderate, even at the great thrust on the stand. The blade is bent backward at the root even at great thrust. It is therefore of interest to know the bending moments when the thrust is small. It is therefore assumed that the propeller runs at the same speed as above (U = 1450 r.p.m.) when it exerts no thrust at all. This is a condition which can easily occur in flying at a low angle of attack and with the engine throttled. If all the aerodynamic forces are put equal to zero, the calculation is greatly simplified, since all the data required to solve the equations have been previously determined. It is only necessary to put all expressions involving aerodynamic forces equal to zero. The calculation is then made exactly as before. Since it affords nothing new of interest, only the results will be given. The stresses are plotted in Figures 24 and 25 and show that, near the blade root, they are considerably greater than in the previous condition, where the aerodynamic forces act. This is also quite natural, for even in the previous loading case, where the aerodynamic forces tended to bend the propeller forward, the opposed "relieving" moments, which are produced by the centrifugal force due to the curved shape of the blades, so far exceed the aerodynamic moment, that the resulting moment is even greater than the moment which the aerodynamic force alone would exert on a straight blade. In normal flight, however, the aerodynamic forces are considerably smaller and they may almost vanish when the airplane is in a glide. The aerodynamic force then produces a backward-bending moment, which is much greater than the aerodynamic forces could produce, even in the most unfavorable case. The blade tends of itself to assume such a position that the effect is minimized. The blade can undergo important deformation, however, only in its outer part, where the cross sections are small. Near the hub no considerable deformation can occur, unless the stresses are very high. The calculated stresses never suffice, however, to cause certain failure, even when they are dangerously high. It must be remembered, however, that the conditions may become considerably more unfavorable through some small material defect, such as may be produced by the weathering of a glued joint. Further considerable stresses may be produced by even small vibrations. The previously mentioned failures occurred mostly in large propellers, which were similar in shape, however, to those investigated here. In so far as known, all the failures occurred in flight, i.e., at low thrust, as might be concluded from the present investigation. The cause of the failures, therefore, was probably excessive curvature of the blades.

Hence great caution must be exercised in trying to give a propeller such a shape that the aerodynamic forces will be counterbalanced by the centrifugal forces. It seems particularly hazardous to let the blade axis near the hub project too far from the plane of the propeller disk. The effect of a faulty shape is not so detrimental in the outer portion of the blade, due to its flexibility, which partially remedies the defect. Near the hub, however, due to the great rigidity, no considerable automatic adjustment is possible, and the moments are greatly affected by the original design. It is impossible to establish any general rule for the design. It may, however, be said that, in the case of a blade having a straight axis, the elastic deformation is always such that the load is actually reduced by the centrifugal force, regardless of all other forces acting on the propeller. This fact was also demonstrated by Reissner. For a curved blade this is not the case. In any event it should be determined whether the load reduction under one operating condition does not involve a load increase under another operating condition.

The propeller blade has thus far been considered as a thin plate. This was accurate enough for the quantities hitherto calculated. This assumption causes the complete disappearance of one factor, which should be considered in many cases. This is the torsional moment exerted on the twisted blade by the longitudinal force. If, for example, we imagine a metal strip twisted about its longitudinal axis so as to form a helical surface, and, if this strip is then subjected to longitudinal tension, it will tend to return to its original flat shape. In order to determine these moments approximately, each fiber may be considered as a twisted blade. When this is pulled, it will tend to untwist. The longitudinal stress in each fiber is known accurately enough from the previous calculation. Due to the twist, the longitudinal direction continually varies along the fiber. There must therefore be everywhere a component perpendicular to the fiber, and this produces a backward turning moment. In thick propellers, such as those of wood, this moment is unimportant, but in very thin blades it may assume great importance and cause considerable distortion. For the approximate calculation of this torsional moment in a thin propeller blade, it is sufficient to take account of the centrifugal force alone (assumed to be uniform throughout the cross section), since, due to its slight rigidity, the blade automatically takes such an attitude that the bending stresses are small. If the angle through which a section is twisted with respect to the preceding section is measured at definite intervals of 10 cm (3.94 in.), for example, this angle is then a criterion for the bending of the individual fibers. The components of the internal forces, producing the torsional moment, can then be readily determined from the change in the direction of the fibers.

Various tests were made in order to determine the deformation of the propellers experimentally. Though all these tests were of a rather primitive nature, it was still possible to verify the calculation with one of the arrangements tested. (Figs. 26 and 27.) The whole contour of a blade of the above mathematically investigated propeller was lined with spark gaps by attaching strips of tin foil 2.5 cm (about an inch) long. One end of this conductor was connected with a condenser by means of a slip ring with brushes, while the other end consisted of a point which passed a stationary point once during every revolution at a distance of about 2 mm (0.08 in.), the latter point being connected with the opposite pole of the condenser. The energy was so adjusted that a spark could pass only when the revolving point was exactly opposite the stationary point. The condenser was charged by an induction coil operated by an alternating current.

First, in order to determine the shape of the propeller in the unstressed condition, the blade was turned to the position where the spark could pass. A photographic camera was then mounted in the plane of the propeller disk in such a way that the axis of the lens was approximately perpendicular to the axis of the propeller blade, and a spark was discharged through the spark gaps and was photographed. The camera was then left in the same position, while the propeller was made to revolve. At 1450 r.p.m. another spark was photographed on the same plate. In order to make sure the camera had not moved, spark gaps were also arranged at two stationary points. In this way Figure 28 was obtained. The deflection can thus be determined with great accuracy. A twisting of the blade, which would be indicated by changes in the distances between the sparks on the leading edge and those on the trailing edge while the propeller is revolving, as compared with the distances between the same sparks when the propeller is at rest, can indeed be detected, but it is so small that it cannot be accurately measured. A twist of one degree would cause a relative displacement of the points of 1.5 to 2.5 mm (0.06 to 0.1 in.) according to the width of the blade at the point investigated. The deflection is readily recognized, however, and agrees well with the above calculation. At the blade tip the measured deflection is 10.6 mm (0.42 in.), and the calculated deflection is 9.7 mm (0.38 in.). The general character of the measured deflection agrees well with the calculated deflection. Even the negative deflection near the hub is recognizable. In order to test the accuracy of this method of measuring, the sparks were photographed with the propeller at rest. The blade was then distorted, and the deflection was accurately measured at several points. Then another photograph was taken and evaluated. It was thus found that deformations of 0.5 mm (0.02 in.) could be determined.

SUMMARY

A method is described for testing the strength of propellers of any shape. It is shown that the shape of the propeller greatly affects the stresses, and that great caution must be exercised if the relieving effect of the centrifugal force, which exists in every propeller with a straight blade axis, is to be increased by curving the blade axis.

In a numerical example, the calculation is made for a certain revolution speed at which the actual distortion is then measured. The results of the calculation agree satisfactorily with the experimental results.

Translation by Dwight M. Miner,
National Advisory Committee
for Aeronautics.

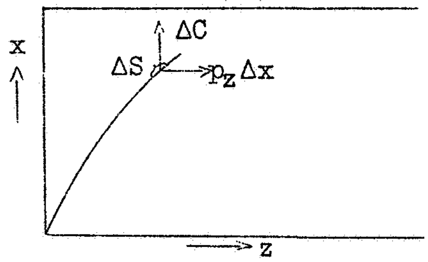

Fig.1 Projection of blade axis on xz plane.

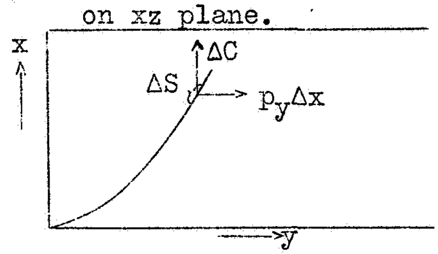

Fig.2 Projection of blade axis on xy plane.

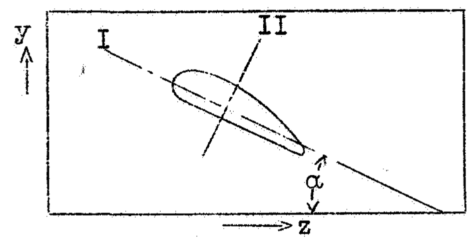

Fig.3 Principal axes and pitch angle of blade section.

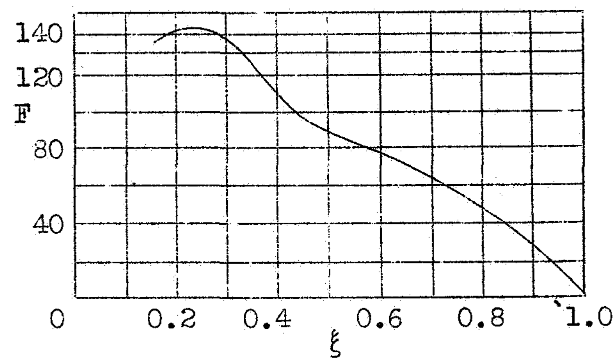

Fig.4 Cross-sectional area plotted against the radius.

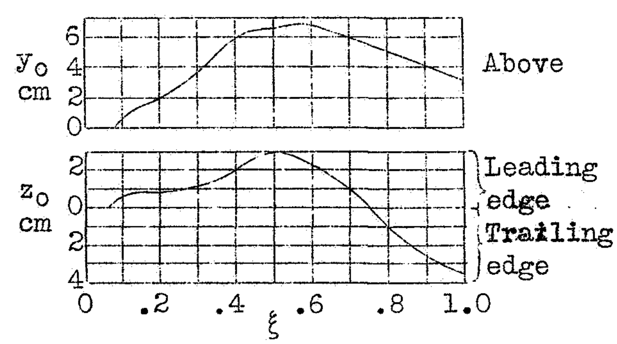

Fig.5 Blade-axis curves.

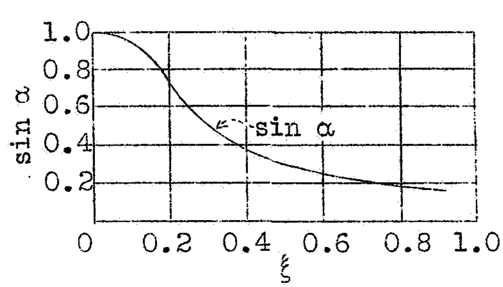

Fig.6a Sine of angle made by chord of blade section with plane of propeller disk.

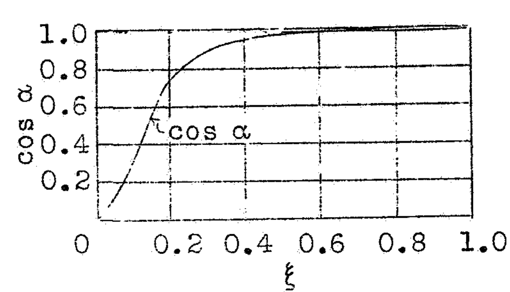

Fig.6b Cosine of angle made by chord of blade section with plane of propeller disk.

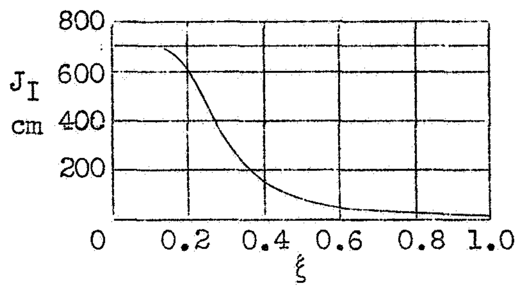

Fig.7 Inertia moments, J_I

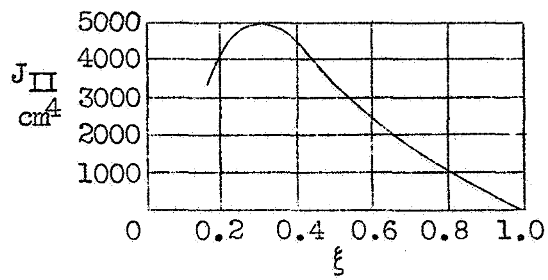

Fig.8 Inertia moments, J_{II}

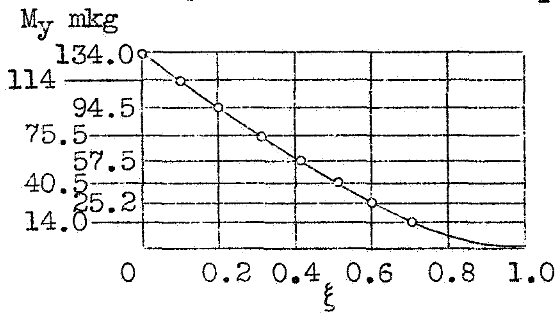

Fig.9 Y component of bending moments due to aerodynamic forces.

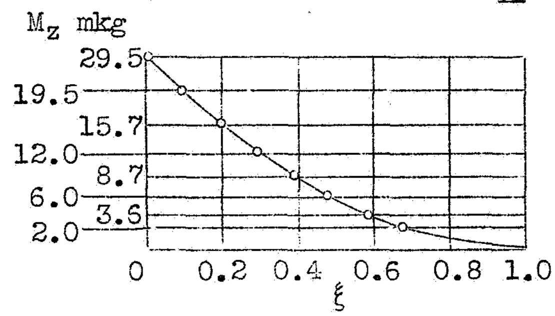

Fig.10 Z component of bending moments due to aerodynamic forces.

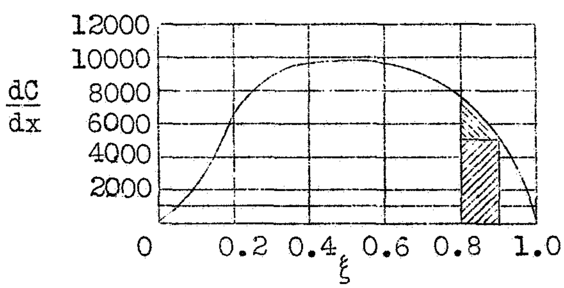

Fig.11 Centrifugal force per unit length of blade.

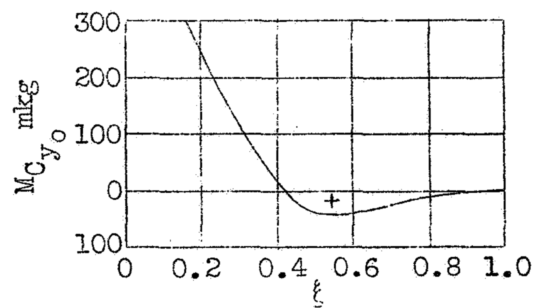

Fig.12

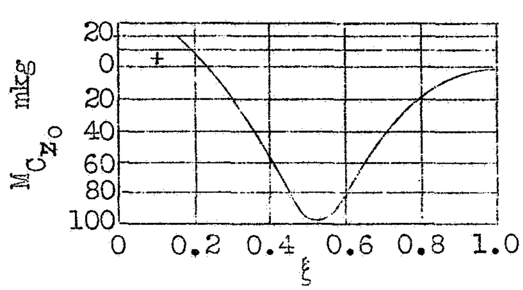

Fig.13

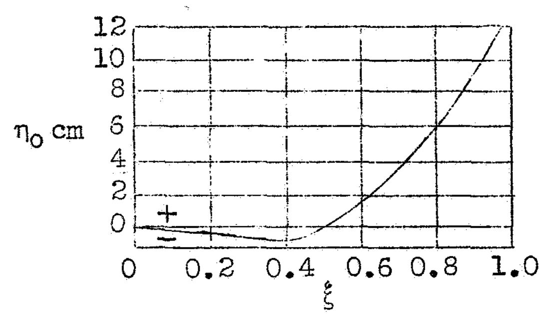

Fig.14

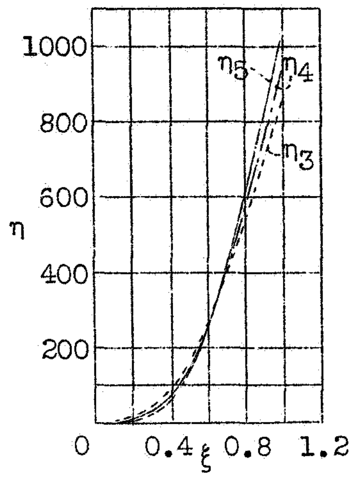

Fig.15

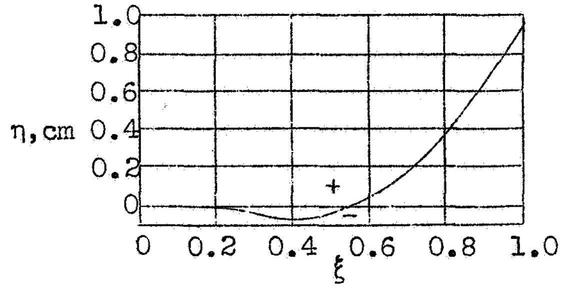

Fig.16 Calculated deflection in direction of flight.

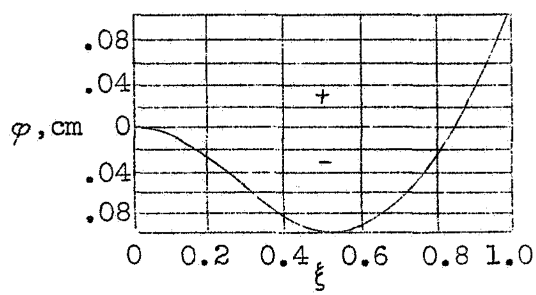

Fig.17 Calculated deflection in plane of propeller disk.

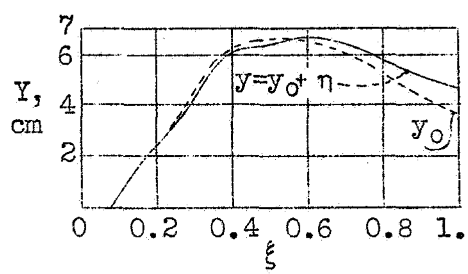

Fig.18 Blade axis with propeller at rest and at full revolution speed on stand.

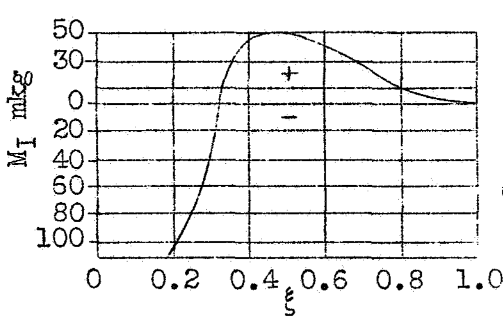

Fig.19 Bending moment, M_I.

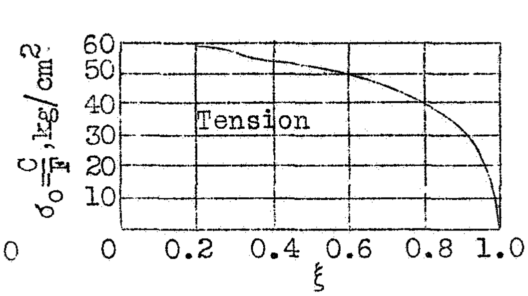

Fig.20 Stress due to centrifugal force.

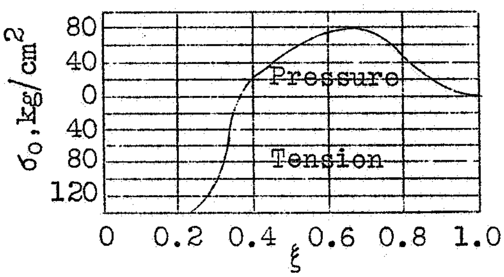

Fig.21 Total stress on pressure side.

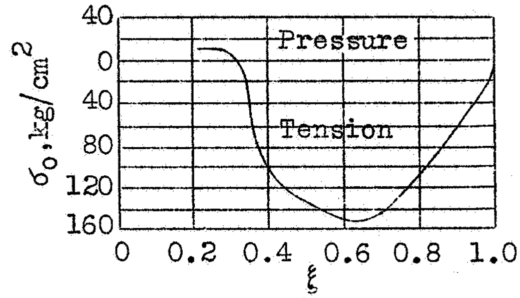

Fig.22 Total stress on tension side.

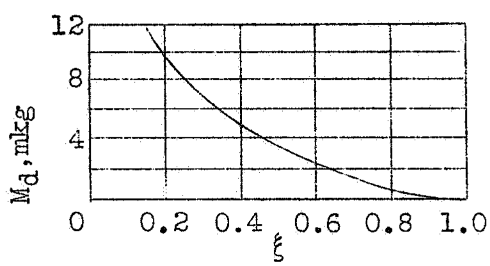

Fig.23 Torsional moment

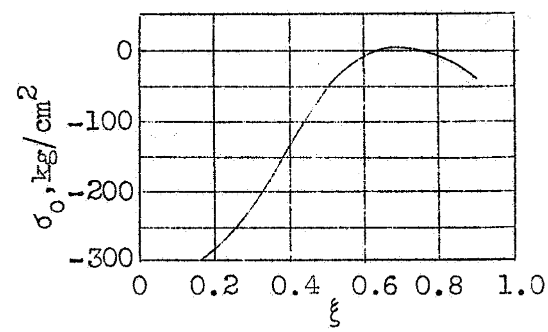

Fig.24 Stress on pressure side.

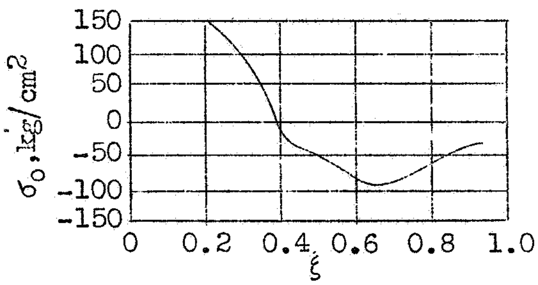

Fig.25 Stress on tension side.

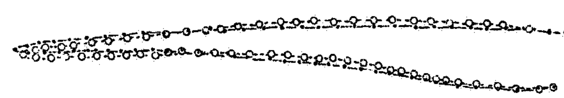

Fig.28 Outline of blade at rest rest and at full speed on stand (copied from photograph for greater clearness).

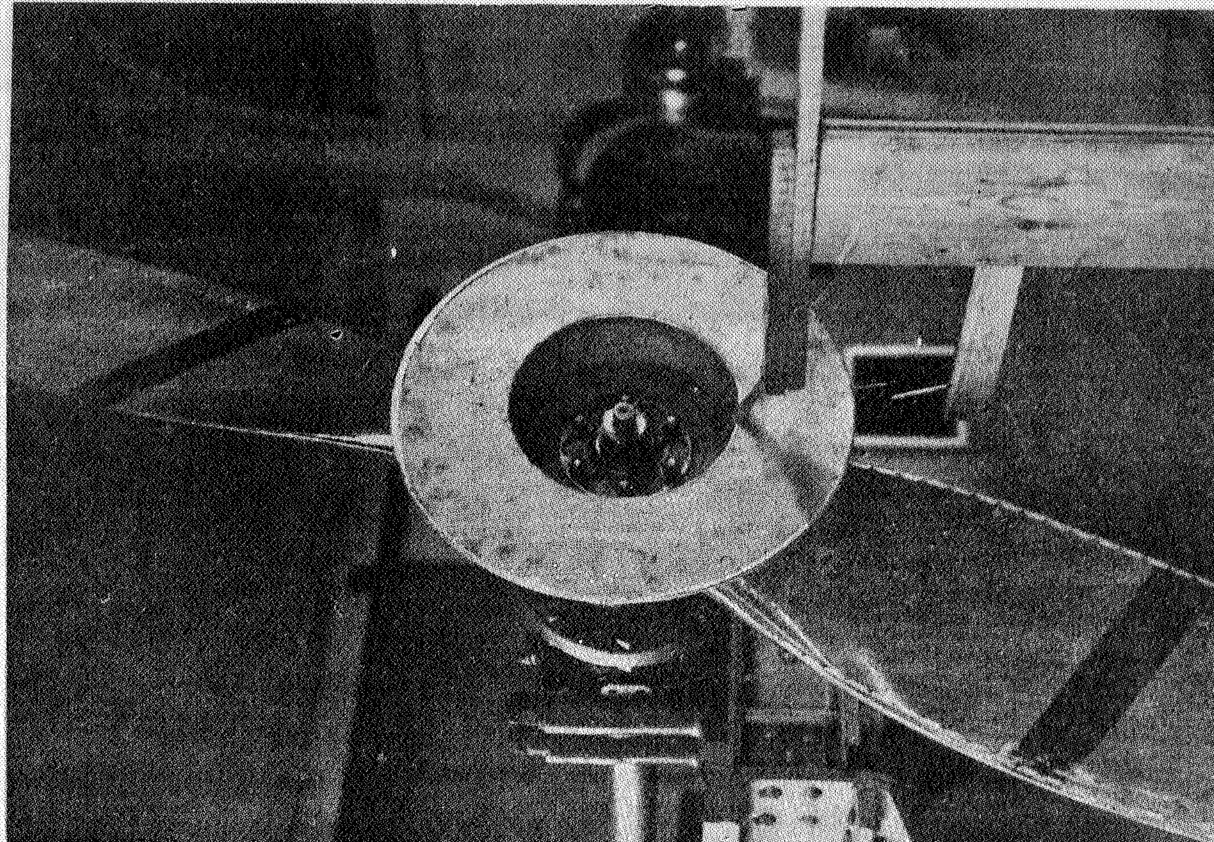

Fig. 27 Close-up of Fig. 26 showing details.

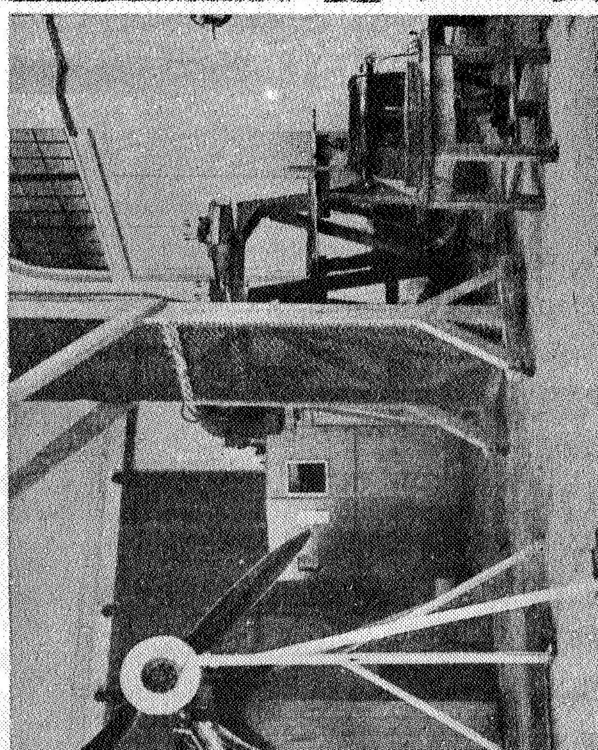

Fig. 26 Experimental arrangement.